Distributed algorithms for in-network recovery of jointly sparse signals

# Distributed algorithms for in-network recovery of jointly sparse signals


**Sophie M. Fosson**[1], **Javier Matamoros**[2], **Carles Antón-Haro**[2], **Enrico Magli**[1]

[1]*Politecnico di Torino - DET, Italy* [2]*CTTC, Barcelona, Spain*



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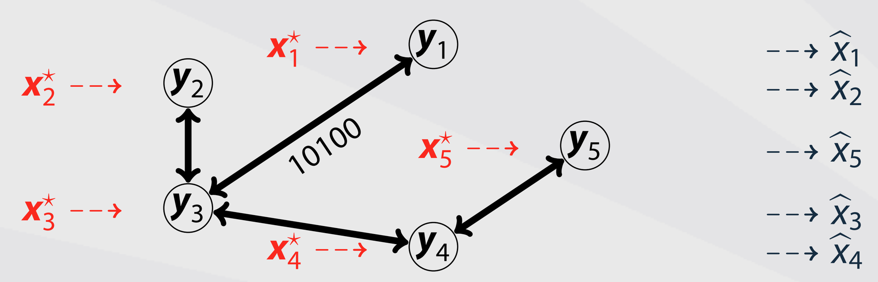
## Problem

*Multiple sparse signals with joint support are individually acquired (and compressed) by sensors of a network. We propose an efficient distributed approach for their reconstruction.*

- Acquisition model:
  - $\mathcal{V} = \{1, 2, \dots, |\mathcal{V}|\}$: set of sensors

  For all $v \in \mathcal{V}$:
  - $\boldsymbol{x}_v^\star \in \mathbb{R}^n$: $k$-sparse signals with joint support $\boldsymbol{s} = \mathbb{1}(\boldsymbol{x}_v^\star) \in \{0,1\}^n$
  - $\boldsymbol{A}_v \in \mathbb{R}^{m\times n}$, $m < n$: sensing matrices
  - $\boldsymbol{y}_v = \boldsymbol{A}_v \boldsymbol{x}_v^\star$: available (compressed) measurements
- Network communication constraints:
  - A. local communication between $v$ and neighboring sensors $u \in \mathcal{N}_v$
  - B. messages in $\{1, \dots, n\}$ ($\lfloor \log_2 n \rfloor + 1$ bits per message)
- Goal: each $v \in \mathcal{V}$ seeks to estimate $x_v^\star$, given $y_v$, and exploiting information about support collected from network
- Example of application: spectrum sensing in cognitive radio networks
- Our approach: iterative soft thresholding (IST), with threshold iteratively adapted to information on support




## Algorithm: DJ-IST

- **Initialization**: $x_v(0) = A_v^T y_v$; $c_v = 0, v \in \mathcal{V}$; $\epsilon, \tau, \lambda, \alpha, \beta > 0$; $p \in \mathbb{N}$.
- **Main cycle**: For $t = 0, 1, \dots, T_{stop}$, for each $v \in \mathcal{V}$, $i = 1, \dots, n$
  1. Gradient: $z_v(t) = x_v(t) + \tau A_v^T(y_v - A_v x_v(t))$,
  2. Threshold tuning: $w_{v,i}(t) = \max\{0, \beta - \alpha|x_{v,i}(t)| - \overline{\mathbb{1}(x_{v,i}(t))}\}$
  3. Soft thresholding:

$$x_{v,i}(t{+}1) = \begin{cases} 0 & \text{if } |z_{v,i}(t)| \leq \lambda + w_{v,i}(t), \text{ or if } z_{v,i}(t) = 0 \text{ and } c_{v,i}(t) \geq p \\ z_{v,i}(t) - \operatorname{sgn}(z_{v,i}(t))[\lambda + w_{v,i}(t)] & \text{otherwise} \end{cases}$$

  4. If $x_{v,i}(t+1) = 0$ and $x_{v,i}(t) \neq 0 \Rightarrow c_{v,i} \leftarrow c_{v,i} + 1$
  5. Transmission: if $\mathbb{1}(x_{v,i}(t+1)) \neq \mathbb{1}(x_{v,i}(t)) \Rightarrow v$ transmits index $i$ to $\mathcal{N}_v$
  6. Stop criterium: if $\|x_v(t+1) - x_v(t)\|_2 < \epsilon \Rightarrow v$ stops

## Theoretical results

Cost Functional: $\mathcal{F}(X, W) =$

$$\sum_{v\in\mathcal{V}} \left\{ \tau \|y_v - A_v x_v\|_2^2 + \sum_{i=1}^{n} 2(\lambda + w_{v,i}) \left[ \alpha |x_{v,i}| + \overline{\mathbb{1}(x_{v,i})} \right] + \|\beta \mathbf{1} - w_v\|_2^2 \right\}$$

$X = \{x_v\}_{v\in\mathcal{V}}$, $W = \{w_v\}_{v\in\mathcal{V}} \in \mathbb{R}^{n\times|\mathcal{V}|}$, $w_{v,i} \geq 0$, $\overline{\mathbb{1}(x_{v,i})} = d_v^{-1} \sum_{u\in\mathcal{N}_v} \mathbb{1}(x_{u,i})$, $d_v =$ degree

- $\ell_1$-reweighted Lasso with communication constraints; $\mathbb{1}(x_v)$ substitutes $x_v$ due to constraint B.
- $\|\beta\mathbf{1} - w_v\|_2^2$ promotes larger thresholds $w_{v,i} \in [0, \beta] \rightarrow$sparsity

DJ-IST alternatively performs:

- minimization with respect to $W \Rightarrow \mathcal{F}(X(t), W(t+1)) \leq \mathcal{F}(X(t), W(t))$;
- a modified IST, in which zeros are definitive (due to terms $\mathbb{1}(x_v)$) $\Rightarrow \mathcal{F}(X(t+1), W(t)) \leq \mathcal{F}(X(t), W(t))$.
- At the beginning, $p$ ``pure'' IST steps on $x_v$, to visit more solutions

In conclusion, for any $t \in \mathbb{N}$, $\mathcal{F}(X(t+1), W(t+1)) \leq \mathcal{F}(X(t), W(t))$

$\Rightarrow$ numerical convergence

$\Rightarrow$ support stabilizes

$\Rightarrow$ $X(t)$ converges to a stationary point $\widehat{X}$

## Analysis of numerical results

Comparison with state-of-the-art: DC-OMP 1 and 2 (T. Wimalajeewa and P. Varshney, IEEE Trans. Signal Process. 2014)

- recover joint support (not $x_v^\star$); require knowledge of $k$
- DC-OMP 1: local communication of messages in $\{1, \dots, n\}$
- DC-OMP 2: multihop communication of messages in $\mathbb{R}^n$ at each step

Performance rankings: (see also graphs below)

★ Transmission load: 1. DC-OMP 1; 2. DJ-IST; 3. DC-OMP 2

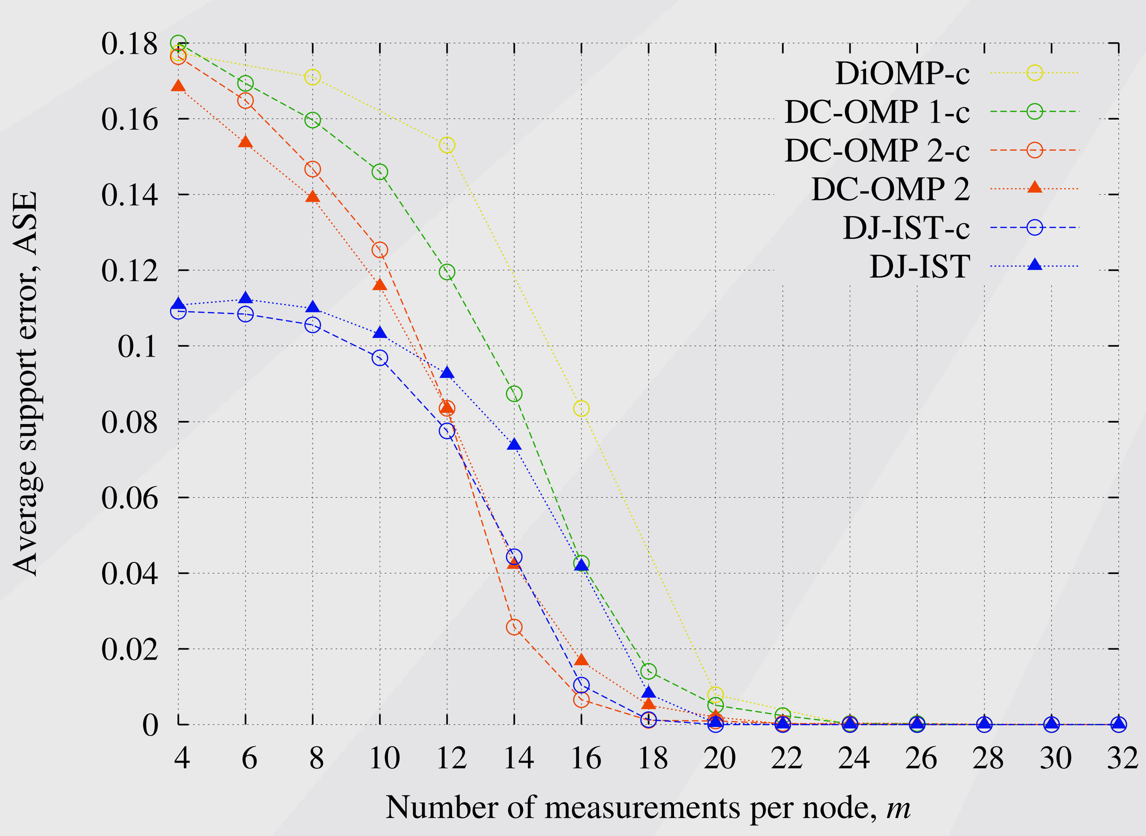
★ Support recovery accuracy: 1. DC-OMP 2; 2. DJ-IST 3. DC-OMP 1

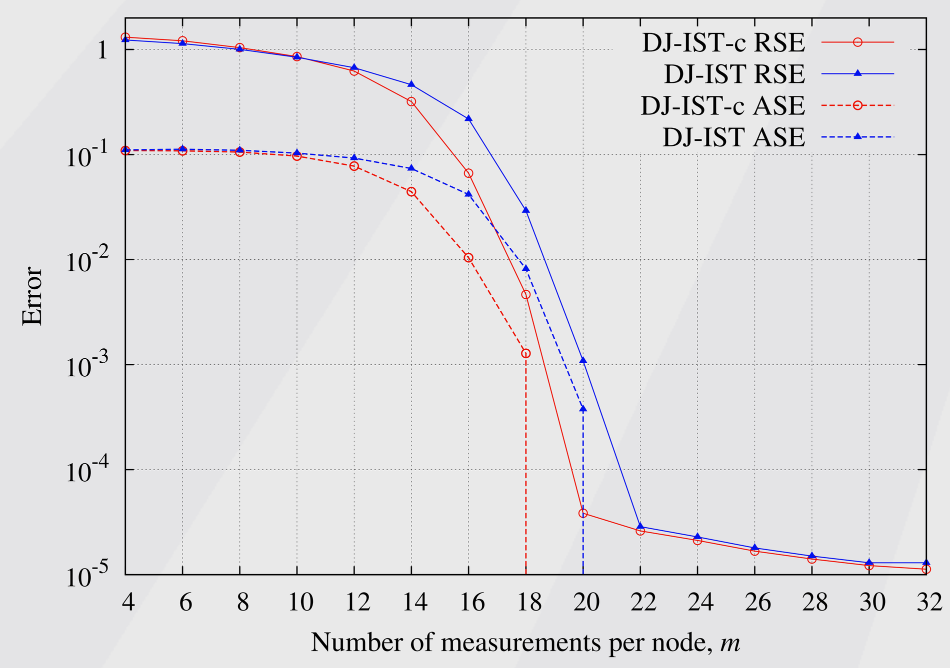

**References**

S.M.F., J.M., C.A.-H., E.M.: A distributed soft thresholding algorithm for jointly sparse signals recovery, submitted, 2015

S.M.F., J.M., C.A.-H., E.M.: Distributed support detection of jointly sparse signals, ICASSP, 2014


## Simulations: $n = 100$, $k = 10$, $|\mathcal{V}| = 10$




## Setting

- support generated uniformly at random; non-zero elements: $\mathcal{N}(0,1)$
- -c $\rightarrow$ complete topology
  other cases: 5-regular topology
- Average support error:

$$\text{ASE} = \frac{1}{|\mathcal{V}|} \sum_{v\in\mathcal{V}} \frac{\|\mathbb{1}(\widehat{x}_v) - \mathbf{s}\|_0}{n}$$

- Relative square error:

$$\text{RSE} = \frac{\sum_{v\in\mathcal{V}} \|x_v^\star - \widehat{x}_v\|_2^2}{\sum_{v\in\mathcal{V}} \|x_v^\star\|_2^2}$$

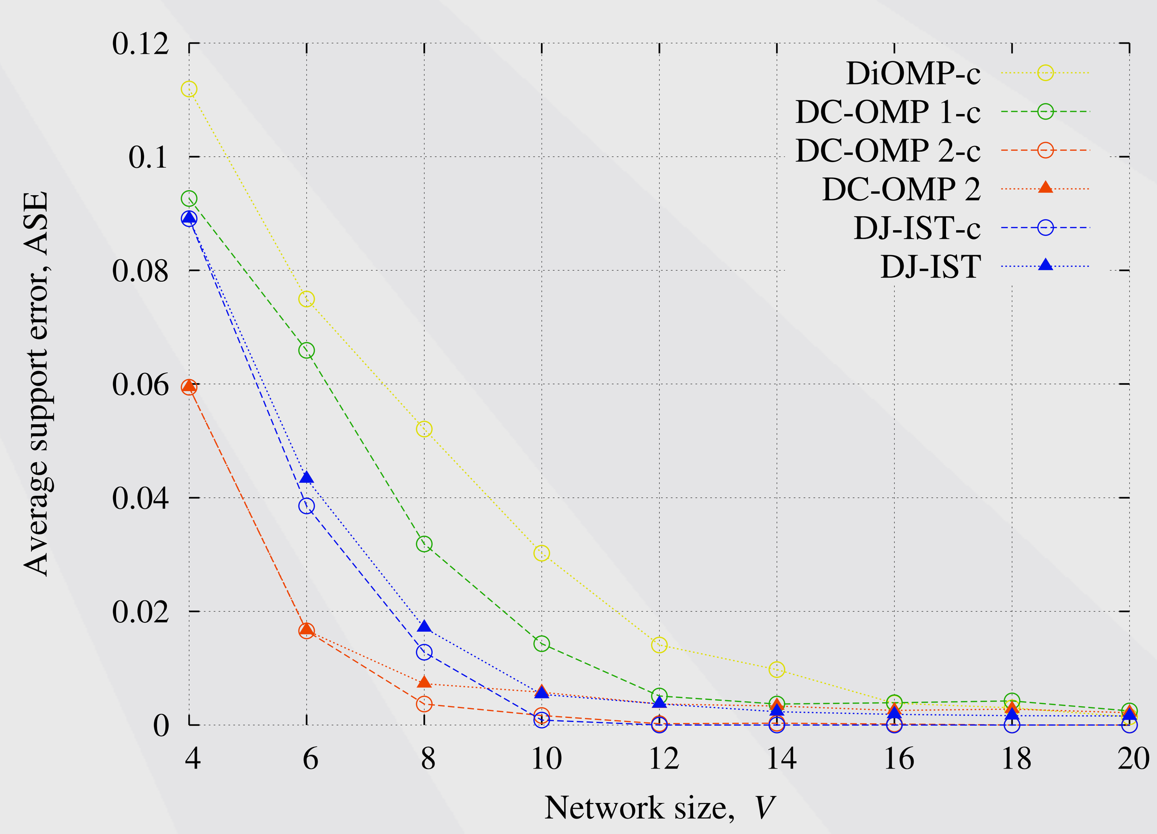
## Total number of transmitted bits

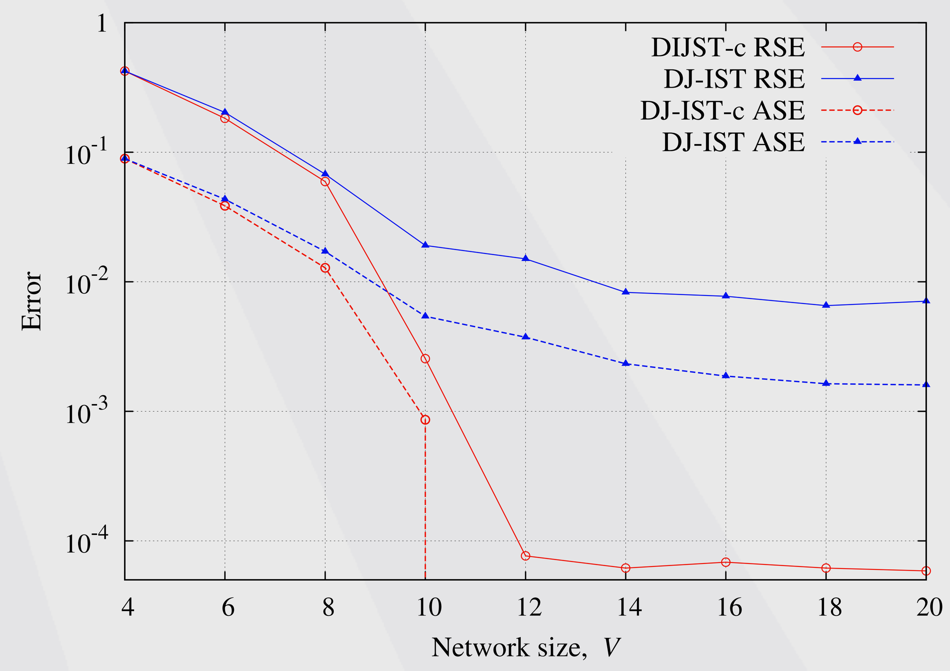
250 runs, $n = 100$, $k = 10$, $|\mathcal{V}| = 10$, $m \in \{4, 6, \dots, 32\}$

(real values quantized over 16 bits)

| Algorithms | Min | Max | Mean |
|---|---|---|---|
| DC-OMP 1 | 840 | 2800 | 1932 |
| DC-OMP 2 | 192945 | 643150 | 463068 |
| DJ-IST | 1924 | 3325 | 2221 |

## Simulations: $n = 100$, $k = 10$, $m = 20$





WEBSITE: www.crisp-erc.eu CONTACTS: sophie.fosson@polito.it